UNSTEADY CONDUCTIVE HEAT TRANSFER FOR BODIES IN AN INFINITE MEDIUM

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The results are presented of Nusselt number calculations for bodies of different geometrical shape located in an infinite medium, the surface temperature of the bodies being functions of time. A number of particular cases are investigated.

In order to solve various practical problems, we require a description of unsteady heat transfer for bodies of different geometrical shape located in an infinite medium.

Where there is conductive heat transfer with the surrounding medium, and a given law of body surface temperature variation with time, the problem in question reduces to the following differential equation:

$$\frac{\partial\Theta}{\partial\tau}=\frac{\partial^2\Theta}{\partial\xi^2}+\frac{n}{\xi}\frac{\partial\Theta}{\partial\xi}. \tag{1}$$

The initial conditions are

$$\tau=0,\ \Theta=0. \tag{2}$$

The boundary conditions are

$$\xi=1,\ \Theta=\varphi_{(\tau)}, \tag{3}$$

$$\xi\to\infty,\ \Theta\to 0. \tag{4}$$

Here $\Theta=(T-T_\infty)/(T_r-T_\infty)$ is temperature; $\tau=at/r^2$ is time; $\xi=x/r$ is a coordinate.

The parameter n, which characterizes the geometry of the system, appears in the problem. With n = 0; 1; 2 we have, respectively, an infinite plate of thickness 2r, a cylinder of infinite length and finite radius, and a sphere.

From the solution of the problem we may then find the heat flux through the body surface ($\xi=1$), while the external heat transfer is described by the Nusselt number

$$\mathrm{Nu}=\frac{\alpha r}{\lambda}=-\frac{1}{\Theta}\frac{\partial\Theta}{\partial\xi}\bigg|_{\xi=1}.$$

This problem has been stated in the literature. For instance, in [1], with the help Karman-type integral relations, an approximate expression is derived for calculating the Nusselt number in the sphere case; this has the form

$$\mathrm{Nu}=1+\frac{1}{\sqrt{3\tau}}\frac{\varphi_{(\tau)}}{\bar\varphi_{(\tau)}},$$

where $\bar\varphi_{(\tau)}=\left(\int_0^\tau\varphi^2_{(z)}\,dz/\tau\right)^{1/2}$.

The method of [1] is very approximate, however, and is used, as will be shown below, only for a bounded class of functions.

In the present paper, for arbitrary type of functions $\varphi(\tau)$ satisfying conditions (2), this problem is solved by the methods of operational calculus.

Leaving out intermediate calculations, the final result has the form

$$n=0,\ \mathrm{Nu}=\frac{1}{\varphi_{(\tau)}}\frac{\partial}{\partial\tau}\int_0^\tau\frac{\varphi_{(z)}}{\sqrt{\pi(\tau-z)}}\,dz, \tag{5}$$

$$n=1,\ \mathrm{Nu}=$$

$$=\frac{4}{\pi^2\varphi_{(\tau)}}\frac{\partial}{\partial\tau}\int_0^\tau\varphi_{(\tau-z)}\int_0^\infty\exp(-zu)\frac{du\,dz}{u\left[I^2_{0(u)}+N^2_{0(u)}\right]}, \tag{6}$$

$$n=2,\ \mathrm{Nu}=1+\frac{1}{\varphi_{(\tau)}}\frac{\partial}{\partial\tau}\int_0^\tau\frac{\varphi_{(z)}}{\sqrt{\pi(\tau-z)}}\,dz, \tag{7}$$

where I_0 and N_0 are zeroth-order Bessel functions of the first and second kind. The integro-differential relations for the Nusselt number are written in the form of Duhamel integrals [2].

By assigning a definite form of functions $\varphi(\tau)$, we may find the Nusselt number from these relations for each particular case.

Let us examine some examples of one of the simplest cases—spherical symmetry (n = 2).

1. Let

$$\varphi_{(\tau)}=k\tau^m, \tag{8}$$

where m is any positive integer. Then, using (7), we have

$$I=\int_0^\tau\frac{(\tau-z)^m}{\sqrt{z}}dz=\tau^{\frac{2m+1}{2}}\left(2-\frac{m}{1!}\cdot\frac{2}{3}+\frac{m(m-1)}{2!}\cdot\frac{2}{5}+\right.$$
$$\left.+\cdots+(-1)^m\frac{2}{2m+1}\right)$$

and

$$\mathrm{Nu}=\frac{(2m+1)}{\sqrt{\pi}}\left[1-\frac{m}{1!}\cdot\frac{1}{3}+\frac{m(m-1)}{2!}\cdot\frac{1}{5}+\right.$$
$$\left.+\cdots+(-1)^m\frac{1}{2m+1}\right]\frac{1}{\sqrt{\tau}}+1. \tag{9}$$

From this it follows that, in the case of a power law of surface temperature variation of type (8), for any value of m, the value of Nu is independent of the coefficient of proportionality k, and $\lim_{\tau\to\infty} \mathrm{Nu} = 1$, i.e., steady heat transfer to or from the surrounding medium is possible. We note that the steady heat transfer region sets in earlier (smaller value of τ), the smaller m is.

At the same time, in a given case, we may obtain, by the method of [1],*

$$\mathrm{Nu}_1 = 1 + \frac{\sqrt{2m+1}}{\sqrt{3}}\,\frac{1}{\sqrt{\tau}}$$

or

$$\Delta = \frac{\mathrm{Nu}-1}{\mathrm{Nu}_1-1} = \frac{\sqrt{2m+1}\,\sqrt{3}}{\sqrt{\pi}} \times$$
$$\times\left[1 - \frac{m}{1!}\cdot\frac{1}{3} + \frac{m(m-1)}{2!}\cdot\frac{1}{5} + \right.$$
$$\left. + \cdots + (-1)^m \frac{1}{2m+1}\right],$$

i.e., the error in calculating Nu by the method of [1] increases with increase of m. Calculations show that even for a square law of surface temperature variation (m = 2), $\Delta = 55\%$.

2. Let us put

$$\varphi_{(\tau)} = A \sin\omega\tau, \tag{10}$$

i.e., the body surface temperature contains harmonic oscillations of amplitude A and frequency ω. Then the integral

$$I = \int_0^\tau \frac{A\sin\omega z}{\sqrt{\pi(\tau - z)}}\,dz = A\sqrt{\frac{2}{\omega}}\left[\sin\omega\tau\cdot C_{(\omega\tau)} - \cos\omega\tau\cdot S_{(\omega\tau)}\right],$$

where $C_{(\omega\tau)} = \sqrt{2/\pi}\int_0^{\sqrt{\omega\tau}} \cos t^2\,dt$ and $S_{(\omega\tau)} = \sqrt{2/\pi} \times \int_0^{\sqrt{\omega\tau}} \sin t^2\,dt$ are Fresnel integrals.

The heat flux through the body surface $\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=1}$ is in this case $\left(\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=1} = -\mathrm{Nu}\Theta\right)$

$$\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=1} = -A\left[\sin\omega\tau + \sqrt{2\omega}\times\right.$$
$$\left.\times\left(\cos\omega\tau\cdot C_{(\omega\tau)} + \sin\omega\tau\cdot S_{(\omega\tau)}\right)\right]. \tag{11}$$

For steady oscillations ($\omega\tau \gg 1$) we may put $S_{(\omega\tau)} = C_{(\omega\tau)} = 1/2$, and (11) takes the form

$$\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=1} = -A_1\sin(\omega\tau + \varphi), \tag{12}$$

where $A_1 = A\sqrt{1 + \sqrt{2\omega} + \omega}$ and $\varphi = \operatorname{arc\,tg}(1 + \sqrt{2/\omega})$.

When $\omega \gg 2$ arc tg $(1 + \sqrt{2/\omega}) \simeq \pi/4$ and $\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=1} = -A_1\sin\left(\frac{\pi}{4} + \omega\tau\right)$.

Thus, in the case of sinusoidal oscillations of body surface temperature, the heat flux also contains harmonic oscillations, but with amplitude A and phase shift φ, dependent on frequency of oscillation ω, i.e., in this case steady heat transfer is impossible.

The Nu number for arbitrary time will have the form

$$\mathrm{Nu} = 1 + \sqrt{2\omega}\,[S_{(\omega\tau)} + \operatorname{ctg}\omega\tau\cdot C_{(\omega\tau)}]. \tag{13}$$

The value of Nu_1 proves to be

$$\mathrm{Nu}_1 = 1 + \frac{2\omega\sin\omega\tau}{\sqrt{3(2\omega\tau - \sin 2\omega\tau)}}, \text{ or } \lim_{\omega\tau\to\infty}\mathrm{Nu}_1 = 1,$$

i.e., in this case the method of [1] gives a result that is wrong in principle.

3. Let

$$\varphi_{(\tau)} = \exp\tau - 1. \tag{14}$$

Then

$$I = \int_0^\tau \frac{\exp z - 1}{\sqrt{\tau - z}}\,dz = -2\sqrt{\tau} + \sqrt{\pi}\exp\tau\cdot\operatorname{erf}(\sqrt{\tau}),$$

where $\operatorname{erf}(\sqrt{\tau}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{\tau}}\exp(-t^2)\,dt$ is the error integral.

The Nu number is

$$\mathrm{Nu} = 1 + \frac{\exp\tau}{\exp\tau - 1}\operatorname{erf}(\sqrt{\tau}), \tag{15}$$

or $\lim_{\tau\to\infty}\mathrm{Nu} = 2$.

Thus, in the case of a power law of type (14) for body surface temperature variation, the heat flux, in the limit as $\tau\to\infty$, is $\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=1} = -2\Theta$, i.e., its intensity is twice as large as in steady heat transfer.

By the method of [1] we obtain

$$\mathrm{Nu}_1 = 1 + \sqrt{\frac{2}{3}}\,\frac{\exp\tau - 1}{\sqrt{(\exp\tau - 2)^2 + \tau - 1}} \text{ and } \lim_{\tau\to\infty}\mathrm{Nu}_1 =$$
$$= 1 + \sqrt{\frac{2}{3}} = 1.815,$$

i.e., the error of this method is insignificant (~9%) only at large values of time; at small times it may be appreciable.

*Nu_1 is the value of the parameter calculated by the method of [1].

NOTATION

T — variable temperature; x — coordinate; t — time; T_∞ — value of temperature when $x \to \infty$; r — characteristic body dimension; T_r — temperature on body surface; λ — thermal conductivity; a — thermal diffusivity; α — external heat transfer coefficient.

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